SOME ASYMPTOTIC RELATIONS IN THE STATISTICAL THEORY OF STABILITY OF SHELLS

(NEKOTORYE ASIMPTOTICHESKIE SOOTNOSHENIIA V STATISTICHESKOI TEORII USTOICHIVOSTI OBOLOCHEK)

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In a report at the All Union conference on the theory and applications of thin shells (Tartu, 1957) the author has proposed an approach to the statistical theory of stability of shells. The basic features of that approach are presented in [1].

The investigations are based on the assumption that all the factors which determine the random character of bending of a shell are divided into three groups:

1) the scatter of the elastic, geometric parameters of the shell, the parameters which determine the manner in which the shell is supported.

It is assumed that the parameters of this group $a_1, \ldots, a_n$ do not depend on time, and that a cumulative law of their distribution $\varphi(a_1, \ldots, a_n)$ is given. The random components of the external forces, constant in time, can also be included here;

2) the continuous random part of the external loads (for simplicity we assume that only the normal component of the external loads is present), which we approximate by the relation

$$Z^{(3)}(P, t) = \sum_{k=1}^{n} \sum_{l=1}^{n_1} a_{kl} \chi_k(P) \psi_l(t) \qquad (1)$$

In this formula $\chi_k(P)$ and ψ_1 are certain determinate functions of the coordinates of time, $\chi_k(P)$ forming a base-system in the "energy" space [2]. We will assume that the random process $Z^{(3)}(P, t)$ is defined by the law of distribution $\theta(a_{k1})$ of random quantities a_{k1};

3) the random part of the external loads, causing the accelerations

of the type which occur in the Brownian motion.

We will look for an approximate solution to the problem of bending of a plate in the form

$$W = \sum_{k=1}^{n} q_k(t)\, \chi_k(P) \tag{2}$$

and we will determine $q(t)$ by the method of Bubnov-Galerkin. In this case, under the assumption of independence of all three groups of factors, the following formula has been obtained for the law of distribution of random quantities q_k, valid for sufficiently large

$$F(q_1, \ldots, q_n, t) = \int_{-\infty}^{\infty} f(q_1, \ldots, q_n, t, a_k, a_{kl})\, \varphi(a_k)\, \theta(a_{kl})\, da_k da_{kl} \tag{3}$$

where f is determined from the Smolukhovskii equation [3].

If a steady-state distribution is considered, it is given by the relation

$$F(q) = \int_{-\infty}^{\infty} \frac{1}{J} e^{-\mu^2 U(q, a_k)} \varphi(a_k)\, da_k \qquad \left(J = \int_{-\infty}^{\infty} {}^{-\mu^2 U(s, a_k)}\, ds\right) \tag{4}$$

Here $U(q, a_k)$ is the potential energy of the system shell-determinate part of the external forces, μ^2 is a parameter, J is the normalizing factor. Regarding the statistical and mechanical meaning of the parameter μ see [1-4].

A certain assumption is made below. In order to formulate it, let us consider the equations which determine q_k at fixed a_k

$$\partial U / \partial q_k = 0 \tag{5}$$

We assume that from these equations q_k is defined by the relations

$$q_k^{\circ} = A_k(a) \tag{6}$$

which perform a single-valued transformation of the domain of parameters a into q°. Note, that this condition is, in fact, the basis of all the relations given in the article [5], because the scope of assumptions of that paper does not allow for the distinction between the probabilities of different branches, in case the relations (6) were multiple-valued.

Let us further assume that q_k can be expressed single-valuedly from (5), (6) in terms of q_k°, so that

$$a_k = B_k(q^\circ) \tag{7}$$

This assumption is not essential (see comment 2) and is introduced here only to simplify the derivations.

Let it be our objective to obtain the asymptotic expansions of $F(q)$ for large values of μ. First of all we note, that in the formula (4), using the relation (7), we can integrate with respect to q_k°, which leads to the relation

$$F(q) = \int_{-\infty}^{\infty} \frac{1}{J} e^{-\mu^2 U(q, q^\circ)} \varphi(B_k) M(q^\circ) dq^\circ, \quad M = \left| \frac{\partial a}{\partial q^\circ} \right| \tag{8}$$

M is the modulus of the Jacobian transformation from a_k into q_k°. Let us first investigate the asymptotic expansion of J for $\mu \to \infty$. For this purpose we look into the form of the potential energy $U(s, q^\circ)$. In the commonly used nonlinear theories of shells the potential energy is a polynomial of the fourth degree with respect to the generalized coordinates. Also, q° is the point of equilibrium, therefore

$$U(s, q^\circ) = \sum_{i,k=1}^{n} U_{ik}(q^\circ)(s_i - q_i^\circ)(s_k - q_k^\circ) + \sum_{i,j,k=1}^{n} U_{ijk}(q^\circ)(s_i - q_i^\circ)(s_j - q_j^\circ)(s_k - q_k^\circ) + \sum_{i,j,k,l=1}^{n} U_{ijkl}(q^\circ)(s_i - q_i^\circ)(s_j - q_j^\circ)(s_k - q_k^\circ)(s_l - q_l^\circ) \tag{9}$$

Moreover, q° is the only point of equilibrium, hence the function $U(s, q^\circ)$ will have no other extreme values besides q°.

In the integral (4) we change the variables

$$\mu(s_i - q_i^\circ) = t_i, \qquad \mu(q_i^\circ - q_i) = x_i \tag{10}$$

Thus we get

$$U\left(\frac{t+x}{\mu} + q, \ \frac{x}{\mu} + q\right) = \sum_{i,k=1}^{n} U_{ik}\left(\frac{x}{\mu} + q\right)\frac{t_i t_k}{\mu^2} + \sum_{i,j,k=1}^{n} U_{ijk}\left(\frac{x}{\mu} + q\right)\frac{t_i t_j t_k}{\mu^3} + \sum_{i,j,k,l=1}^{n} U_{ijkl}\left(\frac{x}{\mu} + q\right)\frac{t_i t_k t_j t_l}{\mu^4} \tag{11}$$

Assuming that the functions on the right-hand side of (11) are sufficiently smooth, for large values of μ we obtain the following expansion

$$U = \frac{Q_2(q, t)}{\mu^2} + \frac{Q_3(q, x, t)}{\mu^3} + \frac{Q_4(q, x, t)}{\mu^4} + \ldots \tag{12}$$

where

$$Q_2(q, t) = \sum_{i,k=1}^{n} U_{ik}(q \quad {}_i t_k$$

$$Q_3(q, x, t) = \sum_{i,k=1}^{n} t_i t_k \sum_{\alpha_1+\ldots\alpha_n=1} R^{(ik)}_{\alpha_1\ldots\alpha_n} x_1{}_{,\ldots,}{}^{\alpha_1} x_n{}^{\alpha_n} + \sum_{i,j,k=1}^{n} U_{ijk}(q)\, t_i t_j t_k \tag{13}$$

$$Q_4(q, x, t) = \sum_{i,k=1}^{n} t_i t_k \sum_{\alpha_1+\ldots\alpha_n=2} R_{\alpha_1,\ldots,\alpha_n}^{(ik)}(q)\, x_1^{\alpha_1}, \ldots, x^{\alpha_n} + \sum_{i,j,k=1}^{n} t_i t_k t_j \sum_{\alpha_1+\ldots\alpha_n=1} S_{\alpha_1,\ldots,\alpha_n}^{(ijk)} x_1^{\alpha_1}, \ldots, x_n^{\alpha_n} \tag{14}$$

$$Q_p(q, x, t) = \sum_{i,k=1}^{n} t_i t_k \sum_{\alpha_1+\ldots\alpha_n=p-2} R_{\alpha_1,\ldots,\alpha_n}^{(ik)} x_1^{\alpha_1}, \ldots, x_n^{\alpha_n} + \sum_{i,j,k=1}^{n} t_i t_k t_j \sum_{\alpha_1+\ldots\alpha_n=p-2} S_{\alpha_1+\ldots\alpha_n}^{(ijk)} x_1^{\alpha_1}, \ldots, x_n^{\alpha_n} + \sum_{i,j,k,l=1}^{n} t_i t_j t_k t_l \sum_{\alpha_1+\ldots\alpha_n=p-2} T_{\alpha_1,\ldots,\alpha_n}^{(ijkl)} x_1^{\alpha_1}, \ldots, x_n^{\alpha_n} \tag{15}$$

In the Formulas (13) to (15) the quantities R, S, T are expressed in terms of derivatives of U_{ik}, U_{ikj}, U_{ikjl}. The following expansion is obtained directly from the relation (12)

$$J\left(\frac{x}{\mu} + q\right) = \frac{1}{\mu^n} J_0(q) + \frac{1}{\mu^{n+1}} J_1(q, x) + \frac{1}{\mu^{n+2}} J_2(q, x) + \ldots \tag{16}$$

where

$$J_0(q) = \int_{-\infty}^{\infty} e^{-Q_2(q,t)}\, dt, \quad J_1(q, x) = -\int_{-\infty}^{\infty} e^{-Q_2(q,t)} Q_3(q, x, t)\, qt$$

$$J_2(q, x) = \int_{-\infty}^{\infty} e^{-Q_2(q,t)} \left(\frac{Q_3^2}{4} - Q_4\right) dt \text{ etc.} \tag{17}$$

From (16), in turn, follows the expansion

$$J^{-1}\left(\frac{x}{\mu} + q\right) = \frac{\mu^n}{J_0}\left(1 + \frac{K_1}{\mu} + \frac{K_2}{\mu^2} + \ldots\right) \tag{18}$$

$$K_1(q, x) = -\frac{J_1(q,x)}{J_0(q)}, \qquad K_2(q, x) = -\frac{J_2(q, x)}{J_0(q)} + \frac{J_1^2(q, x)}{J_0^2(q)} \tag{19}$$

Now, let us expand $U(q, q^{\circ})$. According to (9) we have

$$U_{\iota}(q, q^{\circ}) = \sum_{i,k=1}^{n} U_{ik}(q^{\circ})(q_i - q_i^{\circ})(q_k - q_k^{\circ}) + \sum_{i,j,k=1}^{n} U_{ijk}(q^{\circ})(q_i - q_i^{\circ})(q_j - q_j^{\circ})\times$$

$$\times(q_k - q_k^{\circ}) + \sum_{i,j,k,l=1}^{n} U_{ijkl}(q^{\circ})(q_i - q_i^{\circ})(q_i - q_i)(q_k - q_k^{\circ})(q_l - q_l^{\circ}) \tag{20}$$

Substituting the second of the relations (10) into (20) we will obtain

$$U(q, q^{\circ}) = \sum_{i,k=1}^{n} U_{ik}\left(\frac{x}{\mu} + q\right)\frac{x_i x_k}{\mu^2} + \sum_{i,j,k=1}^{n} U_{ijk}\left(\frac{x}{\mu} + q\right)\frac{x_i x_j x_k}{\mu^3} +$$

$$+ \sum_{i,j,k,l=1}^{n} U_{ijkl}\left(\frac{x}{\mu} + q\right)\frac{x_i x_j x_k x_l}{\mu^4} \tag{21}$$

From (21) follows

$$U(q, q^{\circ}) = \frac{\Pi_2(q, x)}{\mu^2} + \frac{\Pi_3(q, x)}{\mu^3} + \ldots \tag{22}$$

where

$$\Pi_i(q, x) = Q_i(q, x, t)\,|_{t=x}, \qquad i \geqslant 2 \tag{23}$$

From (22) we obtain

$$e^{-\mu^2 U(q, q^{\circ})} = e^{-\Pi_2(q, x)}\left[1 + \frac{\Phi_1(q, x)}{\mu} + \frac{\Phi_2(q, x)}{\mu^2} + \ldots\right] \tag{24}$$

where

$$\Phi_1(q, x) = -\Pi_3(q, x), \qquad \Phi_2(q, x) = \frac{1}{2}\Pi_3^2 - \Pi_4 \tag{25}$$

Furthermore, if sufficient smoothness of φ and M is assumed, we have

$$M(q^{\circ}) = M\left(\frac{x}{\mu} + q\right) = \sum_{k=0}^{\infty}\frac{1}{\mu^k}\sum_{\alpha_1 + \ldots \alpha_n = K} M_{\alpha_1,\ldots,\alpha_n}(q)\, x_1^{\alpha_1}, \ldots, x_n^{\alpha_n} \tag{26}$$

$$M_{0,0,\ldots,0} = M(q)$$

$$\varphi(B_k) = \varphi\left\{B_1\left(\frac{x}{\mu} + q\right), B_2\left(\frac{x}{\mu} + q\right)\ldots B_n\left(\frac{x}{\mu} + q\right)\right\} = \sum_{k=0}^{\infty}\frac{1}{\mu^k}\varphi_{\alpha_1,\ldots,\alpha_n}(q)\, x_1^{\alpha_1,\ldots,}x_n^{\alpha_n}$$

$$\varphi_{0,\ldots,0} = \varphi(B(q)) \tag{27}$$

In the Formulas (26), (27) the quantities $M_{\alpha_1,\ldots,\alpha_n}$ and $\varphi_{\alpha_1,\ldots,\alpha_n}$ are expressed in terms of the derivatives of M, φ, B_k. These formulas can be conveniently presented in the following form

$$M(q^{\circ}) = M(q) + \sum_{k=1}^{\infty}\frac{M_k(q, x)}{\mu^k}, \qquad \varphi(B(q^{\circ})) = \varphi(B(q)) + \sum_{k=1}^{\infty}\frac{\varphi_k(q, x)}{\mu^k} \tag{28}$$

where M_k and φ_k are certain polynomials with respect to x_i.

After these preliminary expansions we turn to Formula (8) for $F(q)$. In this integral we perform the second of the substitutions shown in (10). As a result we obtain

$$F(q)=\frac{1}{\mu^n}\int_{-\infty}^{\infty} J^{-1}\left(\frac{x}{\mu}+q\right)\varphi\left\{B_1\left(\frac{x}{\mu}+q\right)\cdots B_n\left(\frac{x}{\mu}+q\right)\right\}M\left(\frac{x}{\mu}+q\right)e^{-\Pi_2}\left(1+\frac{\Phi_1}{\mu}+\right.$$
$$\left.+\ldots\right)dx=\frac{1}{J_0(q)}\int_{-\infty}^{\infty} e^{-\Pi_2(q,x)}\left(1+\frac{K_1}{\mu}+\frac{K_2}{\mu_2}+\ldots\right)\left[\varphi(B(q))+\frac{\varphi_1}{\mu}+\ldots\right]\left[M(q)+\right.$$
$$\left.+\frac{M_1}{\mu}+\ldots\right]\left(1+\frac{\Phi_1}{\mu}+\ldots\right)dx \tag{29}$$

Now, we bear in mind that due to (23) the following relation is true

$$\int_{-\infty}^{\infty} e^{-\Pi_2(q,x)}dx=\int_{-\infty}^{\infty} e^{-Q_2(q,t)}dt=J_0(q) \tag{30}$$

From (29), (30) we have directly

$$F(q)=\varphi(B(q))\,M(q)+\sum_{k=1}^{\infty}\frac{F_k(q)}{\mu^k} \tag{31}$$

where

$$F_1(q)=\frac{1}{J_0(q)}\int_{-\infty}^{\infty} e^{-\Pi_2(q,x)}\left[\varphi M(K_1+\Phi_1)+\varphi M_1+\varphi_1 M\right]dx \tag{32}$$

$$F_2(q)=\frac{1}{J_0(q)}\int_{-\infty}^{\infty} e^{-\Pi_2(q,x)}\left[\varphi_2 M+\varphi_1 M_1+\varphi M_2+(K_1+\Phi_1)(\varphi M_1+\varphi_1 M)+\varphi M(K_2+\right.$$
$$\left.+K_1\Phi_1+\Phi_1+\Phi_2)\right]dx \qquad \text{etc.} \tag{33}$$

Formula (31) supplies the desired asymptotic expansion. Knowing the expansion (31) one can find the asymptotic expansions for any other random quantity functionally related to $q_1, \ldots, q_n$.

As can be seen from the derivation of Formula (31), there is a considerable number of quadratures to be taken in finding F_k. However, all these quadratures have the form

$$\int_{-\infty}^{\infty} e^{-A(s_1,\ldots,s_n)}s_1^{\alpha_1},\ldots,s_n^{\alpha_n}\,ds_1,\ldots,ds_n \qquad (\alpha_i\geqslant 0) \tag{34}$$

where A is a positive definite quadratic form and α are integers. It is well known that such quadratures are easily expressible in terms of elementary functions. The expansion (31) may be useful for large values of μ. A rigorous treatment could be given for this expansion, which, however, does not appear to be so important at this stage of development of the theory. Note, that for $\mu = \infty$ we obtain from (31)

$$F(q) = \varphi(B(q))\, M(q) \tag{35}$$

Formula (35) provides the basic relation of the so-called "quasi-static approach" for the use of probability methods in the theory of shells [4].

Indeed, in accordance with this approach, out of all the random factors acting on the shell, only a_k are taken into consideration. The law of distribution of q_k° is determined by the elementary formulas of the theory of probability, which provide the relations between the laws of distribution of functionally interconnected random quantities. In the case under consideration we have for the relations (7)

$$F(q^{\circ})\, dq^{\circ} = \varphi(B(q^{\circ}))\, da_k \tag{36}$$

From (36) we obtain

$$F(q^{\circ}) = \varphi(B(q^{\circ})) \left| \frac{\partial a}{\partial q^{\circ}} \right| = \varphi(B(q^{\circ}))\, M(q^{\circ}) \tag{37}$$

i.e. the zero term of the asymptotic expansion (31).

Thus, the relations of [5] are obtained from Formula (3) after a number of additional simplifying assumptions. This fact, however, is also evident from the general considerations.

Comment 1. In this note we have considered a case in which the number of generalized parameters q is equal to the number of random parameters a. Obviously, one can without any trouble obtain the asymptotic expansion, with corresponding corollaries, for the case in which the number of parameters q is smaller than that of parameters a.

Comment 2. In case a_k turn out to be multiple-valued functions of q_k, the entire derivation of the asymptotic formula remains unchanged and only after the transition from a_k to q_k° in the integral (8) it takes the form

$$F(q) = \int_{-\infty}^{\infty} \frac{1}{J}\, e^{-\mu^2 U(q,q^{\circ})} \sum_{m=1}^{p} \varphi(B_k^{\ m}) \left| \frac{\partial a_m}{\partial q^{\circ}} \right| dq^{\circ} \tag{38}$$

In this formula $B_k^{\ m}$, $\alpha_k^{\ m}$ corresponds to the mth branch, p is the total number of branches for a given q°. Thus, the zero term of the expansion (31) takes, accordingly, the form

$$F_0 = \sum_{m=1}^{p} \varphi(B_k^{\ m}) \left| \frac{\partial a^m}{\partial q^{\circ}} \right| \tag{39}$$

Comment 3. The asymptotic expansion (31) may also be used for the case in which the relations (5) have a multiple-valued solution, but there exists a significant difference in the levels of the potential energy. However, if there are several configurations of equilibrium with the levels of potential energy close to each other, then the expansion (31) will change its form, although in this case it also can be readily obtained.

Comment 4. The method of deriving the asymptotic series used in this paper can also be applied when the question is not one of the distribution of q_k, but of other parameters in the problem.

For instance knowing the distribution (3) or (4), one can find the probability p of a snap-through for the system, for which it is also easy to obtain the asymptotic expansion of the form (28). The zero term of the expansion gives the law of distribution of the upper critical value. This can be seen directly from the fact that if only the scatter of the parameters a_k is considered (as it is done in [5]), then for each set of parameters a_k, the snap-through may take place only when the loads have reached the upper critical value.

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